

NON-WEAKLY AMENABLE BEURLING ALGEBRAS

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ABSTRACT. Weak amenability of a weighted group algebra, or a Beurling algebra, is a long-standing open problem. The commutative case has been extensively investigated and fully characterized. We study the non-commutative case. Given a weight function ω on a locally compact group G , we characterize derivations from $L^1(G, \omega)$ into its dual in terms of certain functions. Then we show that for a locally compact IN group G , if there is a non-zero continuous group homomorphism $\varphi: G \rightarrow \mathbb{C}$ such that $\varphi(x)/\omega(x)\omega(x^{-1})$ is bounded on G , then $L^1(G, \omega)$ is not weakly amenable. Some useful criteria that rule out weak amenability of $L^1(G, \omega)$ are established. Using them we show that for many polynomial type weights the weighted Heisenberg group algebra is not weakly amenable, neither is the weighted $\mathbf{ax} + \mathbf{b}$ group algebra. We further study weighted quotient group algebra $L^1(G/H, \hat{\omega})$, where $\hat{\omega}$ is the canonical weight on G/H induced by ω . We reveal that the kernel of the canonical homomorphism from $L^1(G, \omega)$ to $L^1(G/H, \hat{\omega})$ is complemented. This allows us to obtain some sufficient conditions under which $L^1(G/H, \hat{\omega})$ inherits weak amenability of $L^1(G, \omega)$. We study further weak amenability of Beurling algebras of subgroups. In general, weak amenability of a Beurling algebra does not pass to the Beurling algebra of a subgroup. However, in some circumstances this inheritance can happen. We also give an example to show that weak amenability of both $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ does not ensure weak amenability of $L^1(G, \omega)$.

1. INTRODUCTION

Let G be a locally compact group. As usual, we denote the integral of a function f against a fixed left Haar measure by

$$\int f(x)dx.$$

The group algebra $L^1(G)$ is the Banach algebra consisting of all Haar integrable functions on G with the convolution product and the L^1 -norm

$$\|f\|_1 = \int |f(x)|dx.$$

Two functions in $L^1(G)$ are regarded as the same if they are equal almost everywhere on G with respect to the Haar measure.

A *weight function* on G is a locally bounded positive measurable function $\omega: G \rightarrow \mathbb{R}^+$ that satisfies the submultiplicative inequality

$$\omega(xy) \leq \omega(x)\omega(y) \quad (x, y \in G).$$

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Given a weight ω on G , consider

$$L^1(G, \omega) = \{f : f\omega \in L^1(G)\}.$$

Equipped with the norm

$$\|f\|_\omega := \int_G |f(x)|\omega(x) dx$$

and the convolution product, $L^1(G, \omega)$ becomes a Banach algebra, called a *weighted group algebra* or a *Beurling algebra*. The dual space of $L^1(G, \omega)$ may be identified with

$$L^\infty(G, \frac{1}{\omega}) := \{f : f/\omega \in L^\infty(G)\}$$

whose norm is given by

$$\|f\|_{\infty, 1/\omega} = \operatorname{ess\,sup}_{x \in G} \frac{|f(x)|}{\omega(x)} \quad (f \in L^\infty(G, \frac{1}{\omega})).$$

Obviously, as a Banach space $L^1(G, \omega)$ is isometrically isomorphic to $L^1(G)$. However, as Banach algebras these two are very different. For example, it is well-known that $L^1(G)$ is a typical quantum group algebra [16], while $L^1(G, \omega)$ is usually not, although $L^\infty(G, \frac{1}{\omega})$ is a von Neumann algebra with the product $f \cdot g = \frac{1}{\omega}fg$. In fact, $L^1(G, \omega)$ is not even an F-algebra, unless the weight is trivial (meaning that the weight is multiplicative). We refer to [18] for the relation between quantum groups and F-algebras.

The investigation of $L^1(G, \omega)$ goes back to A. Beurling [3], where $G = \mathbb{R}$ was considered. One may find a good account of elementary theory concerning the general weighted group algebra in [24].

Two weight functions ω_1 and ω_2 on G are called *equivalent* if there are constants $c_1, c_2 > 0$ such that

$$c_1\omega_1(x) \leq \omega_2(x) \leq c_2\omega_1(x)$$

locally almost everywhere on G . It is readily seen that if ω_1 and ω_2 are equivalent weights, then $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isomorphic as Banach algebras. It is well-known that a weight on G is always equivalent to a continuous weight on G (see [28], or [24, Theorem 3.7.5] for a proof; note that in [24] the condition $\omega \geq 1$ is not necessary if we do not require the weighted algebra to be a subalgebra of $L^1(G)$). For this reason, unless otherwise is specified, in this paper we always assume that a weight is continuous.

We are concerned with weak amenability of the Beurling algebra $L^1(G, \omega)$. We refer to [7, 8, 22] for research of other aspects regarding Beurling algebras. Special types of groups have been studied in [12, 20, 29]. Related research concerning weighted Fourier algebras may be found in [19, 21].

Recall that a *derivation* from a Banach algebra A to a Banach A -bimodule X is a linear mapping $D: A \rightarrow X$ satisfying $D(ab) = a \cdot D(b) + D(a) \cdot b$ ($a, b \in A$). For every $x \in X$ the map $a \mapsto a \cdot x - x \cdot a$ is a continuous derivation, called an *inner derivation*. Given a Banach A -bimodule X , its dual space X^* is naturally a Banach A -bimodule (called the dual module of X) with the module actions defined by

$$\langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle, \quad \langle x, f \cdot a \rangle = \langle a \cdot x, f \rangle \quad (a \in A, f \in X^*, x \in X).$$

Following B. E. Johnson [14], we call A *amenable* if every continuous derivation from A into any dual Banach A -bimodule X^* is inner. Johnson showed in [14] that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group. Later

N. Gronbaek showed in [10] that the weighted group algebra $L^1(G, \omega)$ is amenable if and only if G is an amenable group and ω is a diagonally bounded weight, i.e., the function $\omega(x)\omega(x^{-1})$ is bounded on G . The latter conditions actually imply that the weight ω is bounded up to a multiplicative factor. Hence, a nontrivial weighted group algebra is intrinsically not an amenable Banach algebra.

Weak amenability for commutative Banach algebras was introduced by Bade, Curtis, and Dales in [2]. Based on a characterization result of [2], Johnson later called a general Banach algebra A *weakly amenable* if every continuous derivation from A into A^* is inner. He showed in [15] that $L^1(G)$ is weakly amenable for all locally compact groups G .

Weak amenability of Beurling algebras has been studied by many authors. In [2] it was shown that $L^1(\mathbb{Z}, \omega_\alpha)$ for the additive group \mathbb{Z} and the polynomial weight $\omega_\alpha(x) = (1 + |x|)^\alpha$ is weakly amenable if and only if $0 \leq \alpha < 1/2$. The same conclusion holds if \mathbb{Z} is replaced with \mathbb{R} ([5, 25, 30]). In [11] N. Gronbaek showed that the Beurling algebra of a commutative discrete group G is weakly amenable if and only if every non-trivial group homomorphism $\Phi: G \rightarrow \mathbb{C}$ satisfies

$$(1) \quad \sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \infty.$$

It turns out that this characterization is still valid for a general commutative locally compact group.

Theorem 1.1. [30, Theorem 3.1] *Let G be an Abelian locally compact group, and ω be a weight on G . The Beurling algebra $L^1(G, \omega)$ is weakly amenable if and only if (1) holds for every continuous non-zero group homomorphism $\Phi: G \rightarrow \mathbb{C}$.*

However, condition (1) is far from being sufficient for $L^1(G, \omega)$ to be weakly amenable if the group G is not commutative. A counterexample associated to discrete $SL_2(\mathbb{R})$ was obtained in [4]. In [26] the first author showed that with a non-trivial polynomial weight ω_α the algebra $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is never weakly amenable. This contrasts with the results on commutative groups \mathbb{Z} and \mathbb{R} mentioned above. Similar investigations concerning the discrete $\mathbf{ax} + \mathbf{b}$ group were also conducted there. Overall, weak amenability of a non-commutative Beurling algebra is still very unclear. So far we have not even seen a non-trivial example of weakly amenable Beurling algebra which is not commutative. The related problem of weak amenability of the center algebra of a Beurling algebra has been studied in [1, 27, 30].

In this paper, in Section 2 we first characterize continuous derivations from $L^1(G, \omega)$ into its dual in terms of certain functions from $L^\infty(G \times G, \frac{1}{\omega \times \omega})$. We then show that the necessity part of Theorem 1.1 remains true if G is an IN group, improving a result of [30]. We further establish a criterion that rules out weak amenability of a Beurling algebra. As an application, we show that the weighted group algebra of the topological Heisenberg group with certain type of “polynomial weights” is not weakly amenable.

In Section 3 we continue the investigation of [26] on weighted $\mathbf{ax} + \mathbf{b}$ group algebras. For the topological $\mathbf{ax} + \mathbf{b}$ group, we show that the Beurling algebra on $\mathbf{ax} + \mathbf{b}$ with a polynomial weight is never weakly amenable. For the discrete case we show that if the weight is independent of b , then the corresponding Beurling algebra is weakly amenable only when the weight is diagonally bounded. This provides us with an example of a locally compact group G with a closed normal subgroup H and a weight ω such that both Beurling algebras $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ are

weakly amenable, but $L^1(G, \omega)$ is not weakly amenable, where $\hat{\omega}$ is a weight on G/H naturally induced from ω .

In Section 4 we study Beurling algebras associated to quotient groups. If H is a closed normal subgroup of G then

$$L^1(G/H, \hat{\omega}) \cong L^1(G, \omega) / J_\omega(G, H),$$

where $J_\omega(G, H)$ is a closed ideal of $L^1(G, \omega)$. We show that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$. This allows us to establish a sufficient condition under which weak amenability of $L^1(G, \omega)$ is inherited by $L^1(G/H, \hat{\omega})$. Using this result, we prove that weak amenability of the tensor product $L^1(G_1, \omega_1) \hat{\otimes} L^1(G_2, \omega_2)$ implies weak amenability of both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$, provided the weights ω_1, ω_2 are bounded away from zero. The question whether the converse is true remains open except for the case when G is Abelian [30, Corollary 3.10]. We also improve a result of [17] concerning weak amenability of a complemented subalgebra.

In Section 5 we investigate Beurling algebras of subgroups. Example 5.1 shows that, even in the Abelian case, weak amenability of a Beurling algebra does not imply weak amenability of the induced Beurling algebra of a subgroup. However, the implication is true under some circumstances. We also investigate the problem of extending a group homomorphism from a subgroup to the whole group in Section 5.

2. CRITERIA RULING OUT WEAK AMENABILITY OF $L^1(G, \omega)$

We start from a characterization of a bounded derivation from $L^1(G, \omega)$ into its dual $L^1(G, \omega)^* = L^\infty(G, \frac{1}{\omega})$. It generalizes a result of B. E. Johnson [15] which deals with the case $\omega \equiv 1$.

Let G_1 and G_2 be two locally compact groups and ω_i be a weight on G_i ($i = 1, 2$). We denote by $\omega_1 \times \omega_2$ the weight on $G_1 \times G_2$ defined by

$$(\omega_1 \times \omega_2)(x_1, x_2) = \omega_1(x_1)\omega_2(x_2) \quad (x_1 \in G_1, x_2 \in G_2).$$

Lemma 2.1. *Let G be a locally compact group and ω be a weight on G . Then for every bounded derivation $D : L^1(G, \omega) \rightarrow L^\infty(G, \frac{1}{\omega})$ there exists a function $\alpha \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ such that*

$$(2) \quad \alpha(xy, z) = \alpha(x, yz) + \alpha(y, zx) \quad (\text{locally a.e. } (x, y, z) \in G \times G \times G) \quad \text{and}$$

$$(3) \quad \langle g, D(f) \rangle = \iint_{G \times G} \alpha(x, y) f(x) g(y) dx dy \quad (f, g \in L^1(G, \omega)).$$

Conversely, every function $\alpha \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ satisfying (2) defines a bounded derivation $D : L^1(G, \omega) \rightarrow L^\infty(G, 1/\omega)$ by the formula (3).

Proof. Given a bounded derivation $D : L^1(G, \omega) \rightarrow L^\infty(G, \frac{1}{\omega})$, the map $(f, g) \mapsto \langle g, D(f) \rangle$ is a bilinear functional on $L^1(G, \omega) \times L^1(G, \omega)$ and we have

$$|\langle g, D(f) \rangle| \leq \|D\| \|g\|_\omega \|f\|_\omega.$$

Hence this map defines a bounded linear functional

$$\alpha \in (L^1(G, \omega) \hat{\otimes} L^1(G, \omega))^* = L^\infty\left(G \times G, \frac{1}{\omega \times \omega}\right)$$

by

$$\langle f \otimes g, \alpha \rangle = \langle g, D(f) \rangle \quad (f, g \in L^1(G, \omega)).$$

It follows that relation (3) holds.

Let $\pi: L^\infty(G, \frac{1}{\omega}) \rightarrow L^\infty(G \times G, \frac{1}{\omega \times \omega})$ be the operator defined by

$$\pi(f)(x, y) = f(xy) \quad \left(f \in L^\infty\left(G, \frac{1}{\omega}\right) \right).$$

From [24, Corollary 3.3.32] it is readily seen that $\pi(f) \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ if $f \in L^\infty(G, \frac{1}{\omega})$, and $\|\pi(f)\|_{\infty, 1/(\omega \times \omega)} = \|f\|_{\infty, 1/\omega}$.

Applying $\pi \otimes \text{id}$ to α , where id stands for the identity operator on $L^\infty(G, \frac{1}{\omega})$, we see that the function $\alpha_1(x, y, z) = \alpha(xy, z)$ belongs to $L^\infty(G \times G \times G, \frac{1}{\omega \times \omega \times \omega})$. Similarly, the functions $\alpha_2(x, y, z) = \alpha(x, yz)$ and $\alpha_3(x, y, z) = \alpha(y, zx)$ also belong to $L^\infty(G \times G \times G, \frac{1}{\omega \times \omega \times \omega})$. In order to show that identity (2) holds, it suffices to verify

$$\langle f \otimes g \otimes h, \alpha_1 \rangle = \langle f \otimes g \otimes h, \alpha_2 \rangle + \langle f \otimes g \otimes h, \alpha_3 \rangle.$$

In fact,

$$\begin{aligned} \langle f \otimes g \otimes h, \alpha_1 \rangle &= \int_{G^3} \alpha(xy, z) f(x) g(y) h(z) dx dy dz \\ &= \int_{G^2} \alpha(y, z) (f * g)(y) h(z) dy dz = \langle h, D(f * g) \rangle \\ &= \langle h, f \cdot D(g) + D(f) \cdot g \rangle = \langle h * f, D(g) \rangle + \langle g * h, D(f) \rangle \\ &= \langle f \otimes g \otimes h, \alpha_2 \rangle + \langle f \otimes g \otimes h, \alpha_3 \rangle \end{aligned}$$

for all $f, g, h \in L^1(G, \omega)$. Therefore (2) holds.

The converse can be easily verified by computation. The proof is complete. \square

Recall that a locally compact group G is an IN group if it has a compact neighborhood of the unit element e which is invariant under all inner automorphisms, i.e., if there is a compact neighborhood U of e such that $gUg^{-1} = U$ for all $g \in G$. It was shown in [30, Remark 3.2] by the second author that for an IN group G the necessity part of Theorem 1.1 remains true under some extra condition. We now can remove this condition. Precisely, we have the following.

Theorem 2.2. *Let G be an IN group and ω be a weight on G . Suppose that there exists a non-trivial continuous group homomorphism $\Phi: G \rightarrow \mathbb{C}$ such that*

$$\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.$$

Then $L^1(G, \omega)$ is not weakly amenable.

Proof. We use Φ to construct a continuous non-inner derivation $D: L^1(G, \omega) \rightarrow L^\infty(G, \frac{1}{\omega})$. Let B be an invariant compact neighborhood of e . Define D as in [30, Theorem 3.1] by

$$(4) \quad D(h)(t) = \int_B \Phi(t^{-1}\xi) h(t^{-1}\xi) d\xi \quad (h \in L^1(G, \omega), t \in G).$$

As indicated in [30, Remark 3.2], D is indeed a continuous derivation. Here we show this by using Lemma 2.1. For all $g, h \in L^1(G, \omega)$ we have

$$\langle g, D(h) \rangle = \int_G \int_{t^{-1}B} \Phi(\xi) h(\xi) d\xi g(t) dt = \int_G \int_G \chi_{t^{-1}B}(\xi) \Phi(\xi) h(\xi) g(t) d\xi dt.$$

Let $\alpha(\xi, t) = \chi_B(t\xi)\Phi(\xi)$. Then α is clearly a measurable function on $G \times G$. Also,

$$\begin{aligned} \sup_{(\xi, t) \in G \times G} \frac{|\alpha(\xi, t)|}{\omega(\xi)\omega(t)} &= \sup_{\xi, t \in G} \frac{|\chi_B(t\xi)\Phi(\xi)|}{\omega(\xi)\omega(t)} = \sup_{\xi, t \in G, t\xi \in B} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(t)} \\ &\leq \sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} \cdot \sup_{\xi, t \in G, t\xi \in B} \frac{\omega(\xi^{-1})}{\omega(t)} \\ &\leq \sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} \cdot \sup_{\xi, t \in G, t\xi \in B} \omega((t\xi)^{-1}) < \infty, \end{aligned}$$

since $\sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} < \infty$ and ω is bounded on the compact set B^{-1} as a continuous function. So we have shown that $\alpha \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$. Next we prove that

$$(5) \quad \alpha(xy, z) = \alpha(x, yz) + \alpha(y, zx) \quad (x, y, z \in G).$$

Fix $x, y, z \in G$. Since $yzx = y(zxy)y^{-1}$ and B is invariant under inner automorphisms, we have that $\chi_B(zxy) = \chi_B(yzx)$. Then we can use the fact that Φ is a group homomorphism to obtain

$$\begin{aligned} \alpha(xy, z) &= \chi_B(zxy)\Phi(xy) = \chi_B(zxy)(\Phi(x) + \Phi(y)) = \chi_B(yzx)\Phi(x) + \chi_B(zxy)\Phi(y) \\ &= \alpha(x, yz) + \alpha(y, zx). \end{aligned}$$

So identity (5) is verified. By Lemma 2.1, D is a bounded derivation from $L^1(G, \omega)$ to $L^\infty(G, \frac{1}{\omega})$.

We now show that for every $h \in L^1(G, \omega)$ the function $D(h) \in L^\infty(G, 1/\omega)$ is continuous. Fix any $t_0 \in G$ and let C be a compact neighborhood of t_0 . Let

$$\beta(x) = \begin{cases} \Phi(x)h(x), & x \in C^{-1}B, \\ 0, & x \notin C^{-1}B. \end{cases}$$

Then,

$$D(h)(t) = \int_B \beta(t^{-1}\xi) d\xi = \int_B L_t(\beta)(\xi) d\xi \quad (t \in C),$$

where L_t is the left translation operator. Since Φ is continuous, $C^{-1}B$ is compact, $h \in L^1(G, \omega)$, and ω is bounded away from zero on compact sets, we have that $\beta \in L^1(G)$. Therefore, for $t \in C$ we have:

$$\begin{aligned} |D(h)(t) - D(h)(t_0)| &= \left| \int_B (L_t\beta(\xi) - L_{t_0}\beta(\xi)) d\xi \right| \leq \int_G |L_t\beta(\xi) - L_{t_0}\beta(\xi)| d\xi \\ &= \|L_t\beta - L_{t_0}\beta\|_{L^1(G)} \rightarrow 0 \quad \text{as } t \rightarrow t_0. \end{aligned}$$

Hence, $D(h)$ is continuous at t_0 . Since t_0 was taken arbitrarily, we conclude that $D(h)$ is a continuous function on G for each $h \in L^1(G, \omega)$.

We are now ready to show that D is not an inner derivation. Suppose, to the contrary, that there exists $f \in L^\infty(G, \frac{1}{\omega})$ such that

$$(6) \quad D(h) = f \cdot h - h \cdot f \quad (h \in L^1(G, \omega)).$$

Fix any $t_0 \in G$ and consider $h_0 = \chi_{t_0^{-1}B} \in L^1(G, \omega)$. Then

$$\begin{aligned} D(h_0)(t_0) &= (f \cdot h_0)(t_0) - (h_0 \cdot f)(t_0) = \int_G f(yt_0)h_0(y) dy - \int_G f(t_0y)h_0(y) dy \\ &= \int_{t_0^{-1}B} f(yt_0) dy - \int_{t_0^{-1}B} f(t_0y) dy = \int_{t_0^{-1}Bt_0} f(y) dy - \int_B f(y) dy = 0. \end{aligned}$$

As we have already shown, $D(h_0)$ is a continuous function. It is also standard that $f \cdot h_0 - h_0 \cdot f$ is a continuous function when $f \in L^\infty(G, \frac{1}{\omega})$ (see, for example, [5, Proposition 7.17]). Therefore,

$$0 = D(h_0)(t_0) = \int_B \Phi(t_0^{-1}\xi)h_0(t_0^{-1}\xi) d\xi = \int_B \Phi(t_0^{-1}\xi) d\xi.$$

Since Φ is a homomorphism, we obtain

$$0 = \int_B \Phi(t_0^{-1}\xi) d\xi = \int_B (\Phi(\xi) - \Phi(t_0)) d\xi = \int_B \Phi(\xi) d\xi - \Phi(t_0)\mu(B),$$

which implies that

$$\Phi(t_0) = \frac{\int_B \Phi(\xi) d\xi}{\mu(B)},$$

where μ denotes the Haar measure on G ($\mu(B) > 0$ since B is a neighborhood of identity and thus contains an open subset). Because $t_0 \in G$ was chosen arbitrarily, it follows that Φ is constant on G , which can happen for a homomorphism Φ only if $\Phi \equiv 0$. This contradiction shows that D is not an inner derivation. The proof is complete. \square

Our next result provides another criterion to rule out weak amenability for a Beurling algebra. For the discrete case it was first obtained by Borwick in his Ph.D. thesis [4] (see also [26]), and has been used in [4] and [26] to study weak amenability of Beurling algebras on discrete $SL_2(\mathbb{R})$, \mathbb{F}_2 , and discrete $\mathbf{ax} + \mathbf{b}$ group.

Let G be a group. Recall that the conjugacy class of $x \in G$ is the set $C_x = \{gxg^{-1} : g \in G\}$. Given a subset B of G , we denote

$$C_B = \{gxg^{-1} : g \in G, x \in B\} = \bigcup_{x \in B} C_x$$

and call it the conjugacy class of B .

Theorem 2.3. *Let G be a locally compact group, $B \neq \emptyset$ be an open set in G with compact closure, and ω be a weight on G that is bounded away from zero on C_B , i.e., there is a constant $\delta > 0$ such $\omega(x) \geq \delta$ for $x \in C_B$. Suppose that there exists a measurable function $\psi : G \rightarrow \mathbb{C}$ bounded on B and such that*

$$(7) \quad \operatorname{ess\,sup}_{x,y \in G} \frac{|\psi(xy) - \psi(yx)|}{\omega(x)\omega(y)} < \infty \quad \text{and}$$

$$(8) \quad \operatorname{ess\,sup}_{z \in C_B} \frac{|\psi(z)|}{\omega(z)} = \infty.$$

Then $L^1(G, \omega)$ is not weakly amenable.

Proof. Suppose that ψ is a function satisfying all aforementioned conditions. Then $\Psi(x, y) = \psi(xy) - \psi(yx)$ is measurable on $G \times G$, and condition (7) ensures that $\Psi \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$. Moreover,

$$\begin{aligned} \Psi(xy, z) &= \psi(xyz) - \psi(zxy) = (\psi(xyz) - \psi(yzx)) + (\psi(yzx) - \psi(zxy)) \\ &= \Psi(x, yz) + \Psi(y, zx) \quad (x, y, z \in G). \end{aligned}$$

Then by Lemma 2.1 Ψ defines a continuous derivation $D : L^1(G, \omega) \rightarrow L^\infty(G, \frac{1}{\omega})$ that satisfies

$$\langle g, D(f) \rangle = \int_{G^2} (\psi(xy) - \psi(yx)) f(x) g(y) \, dx dy \quad (f, g \in L^1(G, \omega)).$$

We show that this derivation D is not inner, which will imply that $L^1(G, \omega)$ is not weakly amenable.

Suppose, to the contrary, that D is inner. Then there exists a function $\varphi \in L^\infty(G, \frac{1}{\omega})$ such that

$$D(f) = \varphi \cdot f - f \cdot \varphi \quad (f \in L^1(G, \omega)).$$

It follows that

$$\langle g, D(f) \rangle = \int_{G^2} (\varphi(xy) - \varphi(yx)) f(x) g(y) \, dx dy \quad (f, g \in L^1(G, \omega)).$$

Denote $\Phi(x, y) = \varphi(xy) - \varphi(yx)$. Then $\Phi \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ and

$$\langle f \otimes g, \Psi - \Phi \rangle = 0 \quad (f, g \in L^1(G, \omega)).$$

Therefore, $\Psi = \Phi$ as the elements of $L^\infty(G \times G, \frac{1}{\omega \times \omega})$. We then have

$$\int_{G^2} (\Psi(x, y) - \Phi(x, y)) U(x, y) \, dx dy = 0 \quad (U \in L^1(G \times G, \omega \times \omega)).$$

On the other hand, if U is in $L^1(G \times G, \omega \times \omega)$, then so is the function $\chi_B(xy)U(x, y)$. Hence,

$$\int_{G^2} (\Psi(x, y) - \Phi(x, y)) \chi_B(xy) U(x, y) \, dx dy = 0.$$

In particular, the last equality holds for all U in $C_{00}(G \times G)$, the space of continuous functions with compact support. For any $U \in C_{00}(G \times G)$, let $V(x, y) = U(x, xy)$. It is evident that $V \in C_{00}(G \times G)$. Thus,

$$\begin{aligned} 0 &= \int_{G^2} (\Psi(x, y) - \Phi(x, y)) \chi_B(xy) V(x, y) \, dx dy \\ &= \int_{G \times B} (\Psi(x, x^{-1}y) - \Phi(x, x^{-1}y)) U(x, y) \, dx dy \end{aligned}$$

for all $U \in C_{00}(G \times G)$. Since $C_{00}(G \times G)$ is dense in $L^1(G \times G, \omega \times \omega)$, we have

$$\int_{G \times B} (\Psi(x, x^{-1}y) - \Phi(x, x^{-1}y))U(x, y) dx dy = 0 \quad (U \in L^1(G \times G, \omega \times \omega)).$$

This implies that $\Psi(x, x^{-1}y) - \Phi(x, x^{-1}y) = 0$ locally almost everywhere on $G \times B$, i.e.,

$$\psi(x^{-1}yx) = \psi(y) - \varphi(y) + \varphi(x^{-1}yx) \quad (\text{locally a.e. on } G \times B).$$

Dividing both sides by $\omega(x^{-1}yx)$ and noting that

$$\operatorname{ess\,sup}_{(x,y) \in G \times G} \frac{|\varphi(x^{-1}yx)|}{\omega(x^{-1}yx)} = \|\varphi\|_{\infty, 1/\omega},$$

we obtain

$$\frac{|\psi(x^{-1}yx)|}{\omega(x^{-1}yx)} \leq \frac{\omega(y)\|\varphi\|_{\infty, 1/\omega} + |\psi(y)|}{\omega(x^{-1}yx)} + \|\varphi\|_{\infty, 1/\omega} \quad (\text{locally a.e. } (x, y) \in G \times B).$$

Since $\|\varphi\|_{\infty, 1/\omega} < \infty$, ψ and ω are bounded on B , and ω is bounded away from zero on C_B , we derive

$$\operatorname{ess\,sup}_{(x,y) \in G \times B} \frac{|\psi(x^{-1}yx)|}{\omega(x^{-1}yx)} < \infty,$$

which is a contradiction to condition (8). Therefore, D is not inner. The proof is complete. \square

As an application of Theorem 2.3, let us consider the topological Heisenberg group. Recall that the Heisenberg group G_H is a 3-dimensional Lie group consisting of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \quad (u, v, w \in \mathbb{R}).$$

It is a unimodular locally compact group with the ordinary Euclidean norm topology and the Lebesgue measure of \mathbb{R}^3 as a Haar measure (see [23, Section 12.1.18]). To simplify the notation, we represent the elements of G_H by (u, v, w) so that $G_H = \mathbb{R}^3$ with the product and inverse operations given by

$$(9) \quad (u, v, w)(a, b, c) = (u + a, v + b, w + c + ub), \quad (u, v, w)^{-1} = (-u, -v, uv - w).$$

Proposition 2.4. *Let ω be a weight on G_H of the form*

$$\omega(u, v, w) = W(|u|, |v|) \quad ((u, v, w) \in G_H).$$

Suppose that

$$(10) \quad \lim_{(x,y) \rightarrow \infty} W(x, y) = \infty.$$

Then $L^1(G_H, \omega)$ is not weakly amenable.

Proof. Consider $B = \{(u, v, w) : |u| < 1, |v| < 1, |w| < 1\}$. Then B is an open set in G_H with compact closure. From (9) we have

$$(u, v, w)(a, b, c)(u, v, w)^{-1} = (a, b, c + ub - va).$$

Therefore $C_B = \{(u, v, w) : |u| < 1, |v| < 1, w \in \mathbb{R}\}$. Since $\omega > 0$ is continuous and depends only on the first two variables, it is obviously both bounded and bounded away from zero on C_B . Consider

$$\tilde{\omega}(t) = \inf\{W(u, v) : u \geq 0, v \geq 0, u + v > |t|\}.$$

It is readily seen that $\tilde{\omega}$ is a positive increasing unbounded continuous function on \mathbb{R} and $\tilde{\omega}(-t) = \tilde{\omega}(t)$ ($t \in \mathbb{R}$). Moreover, $\tilde{\omega}$ is a weight on $(\mathbb{R}, +)$. To see this we note that if $u_i, v_i \geq 0$, $t_i \in \mathbb{R}$ and $u_i + v_i > |t_i|$ ($i = 1, 2$), then

$$\tilde{\omega}(t_1 + t_2) \leq W(u_1 + u_2, v_1 + v_2) \leq W(u_1, v_1)W(u_2, v_2).$$

Taking infimum on the right side over all possible (u_1, v_1) and (u_2, v_2) , we derive the desired inequality

$$\tilde{\omega}(t_1 + t_2) \leq \tilde{\omega}(t_1)\tilde{\omega}(t_2) \quad (t_1, t_2 \in \mathbb{R}).$$

Let

$$\psi(u, v, w) = \chi_{C_B}(u, v, w) \ln \tilde{\omega}(w) \quad ((u, v, w) \in G_H),$$

where χ_{C_B} is the characteristic function of C_B . We aim to show that ψ satisfies all the conditions of Theorem 2.3. It is readily seen that ψ is a locally bounded measurable function on G_H which is unbounded on C_B by (10). Since ω is bounded on C_B , it follows that ψ satisfies condition (8). To show that (7) is satisfied, we let $\mathbf{x} = (u, v, w) \in G_H$ and $\mathbf{y} = (a, b, c) \in G_H$. Then $\mathbf{x}\mathbf{y}$ and $\mathbf{y}\mathbf{x}$ belong to the same conjugacy class. If $\mathbf{x}\mathbf{y} \notin C_B$, then $\mathbf{y}\mathbf{x} \notin C_B$ and condition (7) is obviously satisfied. Assume now that $\mathbf{x}\mathbf{y}, \mathbf{y}\mathbf{x} \in C_B$. Then

$$(11) \quad |\psi(\mathbf{x}\mathbf{y}) - \psi(\mathbf{y}\mathbf{x})| = \left| \ln \frac{\tilde{\omega}(w + c + ub)}{\tilde{\omega}(w + c + av)} \right| \leq |\ln \tilde{\omega}(|ub - av|)| = \ln \tilde{\omega}(|ub - av|).$$

To obtain the last inequality, we used symmetry and submultiplicativity of $\tilde{\omega}$ together with the fact that $\tilde{\omega} \geq 1$ as a symmetric weight function. Since $\mathbf{x}\mathbf{y} \in C_B$, we have that $|u + a| < 1$ and $|v + b| < 1$. So,

$$|ub - av| = |(u + a)b - a(v + b)| \leq |a| + |b|.$$

Similarly, $|ub - av| \leq |u| + |v|$. Then the monotonicity of $\tilde{\omega}$ implies

$$\begin{aligned} \ln \tilde{\omega}(|ub - av|) &\leq \frac{1}{2} \ln (\tilde{\omega}(|a| + |b|) \tilde{\omega}(|u| + |v|)) \\ &\leq \frac{1}{2} \ln (W(|a|, |b|) W(|u|, |v|)) = \frac{1}{2} \ln (\omega(\mathbf{x})\omega(\mathbf{y})) \leq \frac{1}{2} \omega(\mathbf{x})\omega(\mathbf{y}). \end{aligned}$$

In the last step we used the fact that $\omega \geq 1$, which is true since ω is a symmetric weight by the assumption. Combining the last inequality with (11), we see that ψ satisfies condition (7). By Theorem 2.3, $L^1(G_H, \omega)$ is not weakly amenable, and the proof is complete. \square

It is readily seen that the function $\omega_\alpha(u, v, w) = (1 + |u| + |v|)^\alpha$ is a weight on G_H satisfying the condition of Proposition 2.4. So we have

Example 2.5. The Beurling algebra $L^1(G_H, \omega_\alpha)$ is not weakly amenable for any $\alpha > 0$.

It is worth to restate Theorem 2.3 for the discrete group case. We will use this discrete version to study weak amenability of $\ell^1(\mathbf{ax} + \mathbf{b}, \omega)$ in Section 3.

Corollary 2.6. *Let G be a discrete group, $B \neq \emptyset$ be a finite set in G , and ω be a weight on G that is bounded away from zero on the conjugacy class C_B . Suppose that there exists a function $\psi : G \rightarrow \mathbb{R}$ and a constant $c > 0$ such that*

$$(12) \quad |\psi(xy) - \psi(yx)| \leq c\omega(x)\omega(y) \quad (x, y \in G) \quad \text{and}$$

$$(13) \quad \sup_{z \in C_B} \frac{|\psi(z)|}{\omega(z)} = \infty.$$

Then $\ell^1(G, \omega)$ is not weakly amenable. □

For a discrete group G , Lemma 2.1 ensures that each bounded derivation $D : \ell^1(G, \omega) \rightarrow \ell^\infty(G, \frac{1}{\omega})$ gives rise to a function $\alpha \in \ell^\infty(G \times G, \frac{1}{\omega \times \omega})$ such that

$$\alpha(xy, z) = \alpha(x, yz) + \alpha(zx, y) \quad \text{and} \quad D(\delta_x)(y) = \alpha(x, y) \quad (x, y, z \in G).$$

With an additional assumption we can derive further that D must be in the form

$$D(\delta_x) = f \cdot \delta_x - \delta_x \cdot f, \quad \text{i.e.,} \quad \alpha(x, y) = f(xy) - f(yx) \quad (x, y \in G)$$

for some function f on G . We note that although $\alpha \in \ell^\infty(G \times G, \frac{1}{\omega \times \omega})$, in general one cannot expect that $f \in \ell^\infty(G, \frac{1}{\omega})$, which happens only when D is an inner derivation.

Lemma 2.7. *Let G be a discrete group, ω be a weight on G , and $D : \ell^1(G, \omega) \rightarrow \ell^\infty(G, \frac{1}{\omega})$ be a bounded derivation. If $D(\delta_x)(y) = 0$ for all commuting elements $x, y \in G$, then there exists a function f on G such that*

$$(14) \quad D(\delta_x)(y) = f(xy) - f(yx) \quad (x, y \in G).$$

Proof. Since every element commutes with the unit e , from our assumption it follows that $D(\delta_x)(e) = D(\delta_e)(x) = 0$ for all $x \in G$. In particular, $D(xy)(e) = 0$, which implies that $D(\delta_x)(y) = -D(\delta_y)(x)$ for all $x, y \in G$.

We note that G is the disjoint union of all conjugacy classes. To construct f we consider each conjugacy class separately. Let $x_0 \in G$ be fixed. Define f on $C_{x_0} = \{yx_0y^{-1} : y \in G\}$ as follows:

$$f(yx_0y^{-1}) = -D(\delta_{x_0y^{-1}})(y) \quad (y \in G).$$

We first clarify that f is well-defined. Suppose that $u \in C_{x_0}$ has two representations $u = yx_0y^{-1} = zx_0z^{-1}$. Then $x_0y^{-1} = y^{-1}zx_0z^{-1}$. Using the derivation identity, we obtain

$$\begin{aligned} D(\delta_{x_0y^{-1}})(y) &= D(\delta_{(y^{-1}z)(x_0z^{-1})})(y) = (D(\delta_{y^{-1}z}) \cdot \delta_{x_0z^{-1}} + \delta_{y^{-1}z} \cdot D(\delta_{x_0z^{-1}}))(y) \\ &= D(\delta_{y^{-1}z})(x_0z^{-1}y) + D(\delta_{x_0z^{-1}})(z). \end{aligned}$$

Since $yx_0y^{-1} = zx_0z^{-1}$, it is readily seen that the elements $y^{-1}z$ and $x_0z^{-1}y$ commute. By assumption, we then have $D(\delta_{y^{-1}z})(x_0z^{-1}y) = 0$. Thus,

$$D(\delta_{x_0y^{-1}})(y) = D(\delta_{x_0z^{-1}})(z).$$

This shows that the function f is well-defined on C_{x_0} , so it is well-defined on the whole G . (Here, of course, the Axiom of Choice is assumed.) We now prove (14).

For any $x, y \in G$ the elements xy and yx belong to the same conjugacy class, say C_{x_0} . Let $xy = ax_0a^{-1}$. Then

$$\begin{aligned} f(xy) &= -D(\delta_{x_0a^{-1}})(a) = D(\delta_a)(x_0a^{-1}), \\ f(yx) &= f(yax_0(ya)^{-1}) = D(\delta_{ya})(x_0a^{-1}y^{-1}) = D(\delta_a)(x_0a^{-1}) + D(\delta_y)(x). \end{aligned}$$

In the last step we used the relation $ax_0a^{-1}y^{-1} = x$. Therefore,

$$f(xy) - f(yx) = -D(\delta_y)(x) = D(\delta_x)(y).$$

The proof is complete. \square

Proposition 2.8. *Let G be a discrete group and ω be a weight on G such that*

$$\sup_{n \in \mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G).$$

Then for every bounded derivation $D : \ell^1(G, \omega) \rightarrow \ell^\infty(G, \frac{1}{\omega})$ there exists a function f on G such that

$$D(\delta_x)(y) = f(xy) - f(yx) \quad (x, y \in G).$$

Proof. Due to Lemma 2.7, it suffices to show that $D(\delta_x)(y) = 0$ for all bounded derivations $D : \ell^1(G, \omega) \rightarrow \ell^\infty(G, \frac{1}{\omega})$ and all commuting elements $x, y \in G$. Suppose, to the contrary, that $xy = yx$ and $D(\delta_x)(y) = c \neq 0$ for some bounded derivation D . Then, by induction, we have

$$(15) \quad D(\delta_{x^n})(yx^{1-n}) = cn \quad (n \in \mathbb{N}).$$

In fact, this is trivial for $n = 1$. Now assume that (15) holds for $n \in \mathbb{N}$. Then

$$\begin{aligned} D(\delta_{x^{n+1}})(yx^{-n}) &= (D(\delta_x) \cdot \delta_{x^n} + \delta_x \cdot D(\delta_{x^n}))(yx^{-n}) \\ &= D(\delta_x)(y) + D(\delta_{x^n})(yx^{1-n}) = c + cn = c(n+1). \end{aligned}$$

So (15) holds for all $n \in \mathbb{N}$. It then follows that

$$\begin{aligned} \|D\| &\geq \sup_{n \in \mathbb{N}} \frac{\|D(\delta_{x^n})\|_{\ell^\infty(G, 1/\omega)}}{\|\delta_{x^n}\|_{\ell^1(G, \omega)}} \geq \sup_{n \in \mathbb{N}} \frac{|D(\delta_{x^n})(yx^{1-n})|}{\omega(yx^{1-n})} \\ &= \sup_{n \in \mathbb{N}} \frac{|c|n}{\omega((yx)x^{-n})\omega(x^n)} \geq \sup_{n \in \mathbb{N}} \frac{|c|n}{\omega(yx)\omega(x^{-n})\omega(x^n)} \\ &= \frac{|c|}{\omega(yx)} \sup_{n \in \mathbb{N}} \frac{n}{\omega(x^{-n})\omega(x^n)} = \infty \end{aligned}$$

due to the condition on ω . This contradicts to the boundedness of D . The proof is complete. \square

Remark 2.9. Taking into account Lemma 2.1, we see that the function f ensured in Lemma 2.7 and Proposition 2.8 satisfies

$$\sup_{x, y \in G} \frac{|f(xy) - f(yx)|}{\omega(x)\omega(y)} < \infty.$$

3. THE AFFINE MOTION GROUP

In this section we consider the $\mathbf{ax} + \mathbf{b}$ group of all affine transformations $x \mapsto ax + b$ of \mathbb{R} with $a > 0$ and $b \in \mathbb{R}$. Precisely, $\mathbf{ax} + \mathbf{b} = \{(a, b) : a \in \mathbb{R}^+, b \in \mathbb{R}\}$ with product and inverse given by

$$(a, b)(c, d) = (ac, ad + b), \quad (a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a}\right) \quad (a, c \in \mathbb{R}^+, b, d \in \mathbb{R}).$$

With the Euclidean metric topology inherited from \mathbb{R}^2 , $\mathbf{ax} + \mathbf{b}$ is a locally compact group whose left Haar measure is $da db/a^2$.

Lets consider the function $\omega_\alpha(a, b) = (1 + a + |b|)^\alpha$ on $\mathbf{ax} + \mathbf{b}$ ($\alpha > 0$). For $(a, b), (c, d) \in \mathbf{ax} + \mathbf{b}$ we have

$$\begin{aligned} \omega_\alpha((a, b)(c, d)) &= \omega_\alpha(ac, ad + b) = (1 + ac + |ad + b|)^\alpha \leq (1 + |b| + ac + a|d|)^\alpha \\ &\leq (1 + a + |b|)^\alpha (1 + c + |d|)^\alpha = \omega_\alpha(a, b)\omega_\alpha(c, d). \end{aligned}$$

This shows that ω_α is indeed a (continuous) weight on $\mathbf{ax} + \mathbf{b}$.

Proposition 3.1. *Let ω_α ($\alpha > 0$) be the weight on $\mathbf{ax} + \mathbf{b}$ defined as above. Then $L^1(\mathbf{ax} + \mathbf{b}, \omega_\alpha)$ is not weakly amenable.*

Proof. Clearly, $\omega_\alpha \geq 1$ on $\mathbf{ax} + \mathbf{b}$. Let $B = \{(a, b) : 1 < a < 2, 1 < b < 2\}$. Then B is open and \overline{B} is compact in $\mathbf{ax} + \mathbf{b}$. Since

$$(c, d)(a, b)(c, d)^{-1} = (ac, bc + d) \left(\frac{1}{c}, -\frac{d}{c}\right) = (a, -ad + bc + d),$$

we have that $C_B = \{(a, b) : 1 < a < 2, b \in \mathbb{R}\}$. Consider the auxiliary function $\Psi : \mathbf{ax} + \mathbf{b} \rightarrow \mathbb{R}^+$ defined by

$$\Psi(a, b) = \begin{cases} \max\{a - 1, |b|\} & \text{if } 1 < a < 2, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, Ψ is a positive measurable function on $\mathbf{ax} + \mathbf{b}$. We show that it also satisfies

$$(16) \quad \frac{\Psi(yz)}{\Psi(zy)} \leq \omega_1(y)\omega_1(z) \quad (y, z \in \mathbf{ax} + \mathbf{b}),$$

where $\omega_1(a, b) = (1 + a + |b|)$. Let $y = (a, b)$, $z = (c, d) \in \mathbf{ax} + \mathbf{b}$. Then $yz = (ac, ad + b)$ and $zy = (ac, bc + d)$. If $0 < ac \leq 1$ or $ac \geq 2$, then $\Psi(yz) = \Psi(zy) = 1$ and hence (16) holds trivially. Now assume $1 < ac < 2$. Then by the definition of Ψ we have

$$ac - 1 \leq \Psi(zy) \leq \Psi(zy)\omega_1(y)\omega_1(z) \quad \text{and}$$

$$\begin{aligned} |ad + b| &= |a(bc + d) - b(ac - 1)| \leq a|bc + d| + |b|(ac - 1) \\ &\leq \max\{ac - 1, |bc + d|\}(a + |b|) = \Psi(zy)(a + |b|) \leq \Psi(zy)\omega_1(y)\omega_1(z). \end{aligned}$$

Thus

$$\Psi(yz) = \max\{ac - 1, |ad + b|\} \leq \Psi(zy)\omega_1(y)\omega_1(z).$$

This shows that (16) still holds if $1 < ac < 2$. Therefore, (16) holds for all $y, z \in \mathbf{ax} + \mathbf{b}$.

We now let $\psi = \ln \Psi$. Clearly, ψ is a measurable function supported on C_B and bounded on B . We show that it also satisfies the conditions

$$(17) \quad \operatorname{ess\,sup}_{z \in C_B} \frac{|\psi(z)|}{\omega_\alpha(z)} = \infty \quad \text{and}$$

$$(18) \quad |\psi(zy) - \psi(yz)| \leq C\omega_\alpha(y)\omega_\alpha(z) \quad (y, z \in \mathbf{ax} + \mathbf{b})$$

for some constant $C > 0$. Indeed,

$$\operatorname{ess\,sup}_{z \in C_B} \frac{|\psi(z)|}{\omega_\alpha(z)} \geq \sup_{1 < a < 2} \frac{|\psi(a, a-1)|}{\omega_\alpha(a, a-1)} = \sup_{1 < a < 2} \frac{|\ln(a-1)|}{(2a)^\alpha} = \infty.$$

So (17) is verified. To show (18) we may assume, without loss of generality, that $\Psi(yz) \geq \Psi(zy)$. Then, using (16), we obtain

$$\begin{aligned} \omega_\alpha(y)\omega_\alpha(z) &= (\omega_1(y)\omega_1(z))^\alpha \geq \left(\frac{\Psi(yz)}{\Psi(zy)}\right)^\alpha \geq \ln\left(\frac{\Psi(yz)}{\Psi(zy)}\right)^\alpha \\ &= \alpha \left| \ln \frac{\Psi(yz)}{\Psi(zy)} \right| = \alpha |\psi(yz) - \psi(zy)|. \end{aligned}$$

It follows that ψ satisfies (18) with $C = 1/\alpha$. Therefore, the function ψ satisfies all the conditions of Theorem 2.3. This shows that $L^1(\mathbf{ax} + \mathbf{b}, \omega_\alpha)$ is not weakly amenable. The proof is complete. \square

We now equip $\mathbf{ax} + \mathbf{b}$ with the discrete topology. It is readily seen that $H_{\mathbf{b}} = \{(1, b) : b \in \mathbb{R}\}$ is a normal subgroup of $\mathbf{ax} + \mathbf{b}$, and $(\mathbf{ax} + \mathbf{b})/H_{\mathbf{b}} \cong (\mathbb{R}^+, \cdot)$ through the group homomorphism $[(a, b)] \mapsto a$.

Proposition 3.2. *Let ω be a weight on $\mathbf{ax} + \mathbf{b}$ that is bounded away from zero and is bounded on $H_{\mathbf{b}}$. Then $\ell^1(\mathbf{ax} + \mathbf{b}, \omega)$ is weakly amenable if and only if ω is diagonally bounded on $\mathbf{ax} + \mathbf{b}$.*

Proof. The sufficiency is due to [26, Proposition 4.1].

For the necessity, we assume that ω is not diagonally bounded. Let $\hat{\omega}$ be the function on $\mathbf{ax} + \mathbf{b}$ defined by $\hat{\omega}(z) = \inf_{h \in H_{\mathbf{b}}} \omega(zh)$. Clearly, $\hat{\omega}$ is submultiplicative

on $\mathbf{ax} + \mathbf{b}$ and $\hat{\omega}(a, b)$ is independent of b . We simply denote $\hat{\omega}(a, b)$ by $\hat{\omega}(a)$. Then $\hat{\omega}$ is a submultiplicative function on \mathbb{R}^+ . It is easy to verify further that

$$(19) \quad \hat{\omega}(a) \leq \omega(a, b) \leq \tilde{c}\hat{\omega}(a) \quad ((a, b) \in \mathbf{ax} + \mathbf{b}),$$

where $\tilde{c} = \sup_{h \in H_{\mathbf{b}}} \omega(h)$. By our assumption $0 < \tilde{c} < \infty$.

Consider the singleton set $B = \{(1, 1)\}$. The conjugacy class of B is

$$C_B = \{y \cdot (1, 1) \cdot y^{-1} : y \in \mathbf{ax} + \mathbf{b}\} = \{(1, b) : b > 0\}.$$

Define $\psi : \mathbf{ax} + \mathbf{b} \rightarrow \mathbb{R}$ by

$$\psi(a, b) = \begin{cases} \ln(\hat{\omega}(b)\hat{\omega}(b^{-1})) & \text{if } a = 1, b > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, ψ vanishes outside the conjugacy class C_B . We show that

$$(20) \quad |\psi(zy) - \psi(yz)| \leq \omega(y)\omega(z) \quad (y, z \in \mathbf{ax} + \mathbf{b}).$$

Note that zy and yz always belong to the same conjugacy class for $y, z \in \mathbf{ax} + \mathbf{b}$. So it suffices to verify (20) for $zy, yz \in C_B$. Let $yz = (1, b)$, and $z = (k, l)$, $b, k > 0$, $l \in \mathbb{R}$. Then

$$y = (yz)z^{-1} = (k^{-1}, (-l + bk)k^{-1}), \quad zy = (1, bk).$$

It follows that

$$|\psi(zy) - \psi(yz)| = |\psi(1, bk) - \psi(1, b)| = \left| \ln \frac{\hat{\omega}(bk)\hat{\omega}((bk)^{-1})}{\hat{\omega}(b)\hat{\omega}(b^{-1})} \right| \leq |\ln(\hat{\omega}(k)\hat{\omega}(k^{-1}))|$$

since

$$\frac{1}{\hat{\omega}(k)\hat{\omega}(k^{-1})} \leq \frac{\hat{\omega}(bk)\hat{\omega}((bk)^{-1})}{\hat{\omega}(b)\hat{\omega}(b^{-1})} \leq \hat{\omega}(k)\hat{\omega}(k^{-1}).$$

But $\hat{\omega}(k)\hat{\omega}(k^{-1}) \geq \hat{\omega}(e) \geq 1$. So $|\ln(\hat{\omega}(k)\hat{\omega}(k^{-1}))| = \ln(\hat{\omega}(k)\hat{\omega}(k^{-1})) \leq \hat{\omega}(k)\hat{\omega}(k^{-1})$, which implies

$$|\psi(zy) - \psi(yz)| \leq \hat{\omega}(k)\hat{\omega}(k^{-1}).$$

On the other hand, relation (19) yields

$$\omega(y) \geq \hat{\omega}(k^{-1}), \quad \omega(z) \geq \hat{\omega}(k).$$

Thus we obtain (20) as desired. Moreover, using (19) again, we have

$$\begin{aligned} \sup_{x \in C_B} \frac{|\psi(x)|}{\omega(x)} &= \sup_{b > 0} \frac{|\psi(1, b)|}{\omega(1, b)} = \sup_{b > 0} \frac{|\ln(\hat{\omega}(b)\hat{\omega}(b^{-1}))|}{\omega(1, b)} \\ &\geq \frac{\sup_{z \in \mathbf{a}\mathbf{x} + \mathbf{b}} |\ln(\omega(z)\omega(z^{-1}))| - |\ln \tilde{c}^2|}{\tilde{c}} = \infty, \end{aligned}$$

since ω is not diagonally bounded on G . From Corollary 2.6, $\ell^1(\mathbf{a}\mathbf{x} + \mathbf{b}, \omega)$ is not weakly amenable. The proof is complete. \square

4. BEURLING ALGEBRA OF QUOTIENT GROUPS

Let G be a locally compact group, ω be a weight on G , and H be a closed normal subgroup of G . Define $\hat{\omega}$ on the quotient group G/H by

$$\hat{\omega}([x]) = \inf_{z \in [x]} \omega(z) = \inf_{\xi \in H} \omega(x\xi),$$

where $[x]$ stands for the coset of x in G/H ($x \in G$). From [13, Theorem 11.0] we know that $\hat{\omega}$ is a nonnegative upper semicontinuous and hence is a locally bounded measurable function on G/H . To avoid $\hat{\omega}$ being trivial, here and in the rest of this section we assume that ω is bounded away from zero. Then $\hat{\omega}$ is a locally bounded measurable weight function on G/H [24, Theorem 3.7.13]. As indicated in Section 1, $\hat{\omega}$ is equivalent to a continuous weight. We note that in studying the weighted group algebra $L^1(G, \omega)$, requiring ω to be bounded away from zero is not really a restriction if G is an amenable group. Indeed, if G is amenable, then by [28, Lemma 1] there exists a continuous positive character $\phi : G \rightarrow (\mathbb{R}^+, \cdot)$ such that $\phi \leq \omega$ on G . Then $\tilde{\omega} = \omega/\phi \geq 1$ is a weight on G and $L^1(G, \omega)$ is isometrically isomorphic to $L^1(G, \tilde{\omega})$ as a Banach algebra.

We are concerned with the relation between weak amenability of $L^1(G, \omega)$ and that of $L^1(G/H, \hat{\omega})$. First, as a simple consequence of Theorems 2.2 and 1.1 we obtain the following.

Proposition 4.1. *Let G be an IN group and H be a closed normal subgroup of G such that G/H is Abelian. Suppose that ω is a weight on G that is bounded away from zero. If $L^1(G, \omega)$ is weakly amenable, then so is $L^1(G/H, \hat{\omega})$.*

Proof. If $L^1(G/H, \hat{\omega})$ were not weakly amenable, according to Theorem 1.1, there would exist a continuous non-trivial group homomorphism $\Phi : G/H \rightarrow \mathbb{C}$ such that

$$\sup_{[x] \in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty.$$

Then the natural extension $\tilde{\Phi}$ of Φ to G defined by $\tilde{\Phi}(x) = \Phi([x])$ ($x \in G$) is a non-trivial continuous group homomorphism from G to \mathbb{C} and

$$\sup_{x \in G} \frac{|\tilde{\Phi}(x)|}{\omega(x)\omega(x^{-1})} \leq \sup_{[x] \in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty,$$

since $\hat{\omega}([x]) \leq \omega(x)$ ($x \in G$). By Theorem 2.2 this implies that $L^1(G, \omega)$ is not weakly amenable, contradicting our assumption. \square

For the general case, according to the theory established in [24], there is a standard Banach algebra homomorphism T from $L^1(G, \omega)$ onto $L^1(G/H, \hat{\omega})$ defined by

$$(21) \quad (Tf)([x]) = \int_H f(xh) dh \quad (f \in L^1(G, \omega), x \in G).$$

The kernel of T is a closed ideal in $L^1(G, \omega)$ and we denote it by $J_\omega(G, H)$. It was proved in [24, Theorem 3.7.13] that T induces an isometric isomorphism between $L^1(G, \omega)/J_\omega(G, H)$ and $L^1(G/H, \hat{\omega})$. So we are in the situation concerned by the following well-known result.

Proposition 4.2. [9, Proposition 2.4] *Let A be a weakly amenable Banach algebra and I be a closed ideal in A . Then A/I is weakly amenable if and only if I has the trace extension property as described in the following.*

For every $\lambda \in I^$ satisfying $a \cdot \lambda = \lambda \cdot a$ ($a \in A$), there is a $\tau \in A^*$ such that $\tau|_I = \lambda$ and $\tau(ab) = \tau(ba)$ ($a, b \in A$).*

We now investigate when $J_\omega(G, H)$ has the trace extension property as a closed ideal of $L^1(G, \omega)$. We start from proving that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$ as a Banach subspace. For this we need two technical lemmas.

Lemma 4.3. [24, Proposition 8.1.16] *Let H be a closed subgroup of a locally compact group G and U be a non-empty open set in G with compact closure. Then there is a subset Y of G such that the family $\{UyH\}_{y \in Y}$ covers G and is locally finite, i.e., every point of G has a neighborhood intersecting at most finitely many members of the family.*

The second lemma we need generalizes the investigation in [24, Section 8.1] of the Bruhat function associated to a normal subgroup.

Lemma 4.4. *Let G be a locally compact group, H be a closed normal subgroup of G , and ω be a weight on G bounded away from zero. Then there exists a continuous function $g \geq 0$ on G and a constant $c > 0$ such that the following two conditions are satisfied:*

$$(22) \quad \int_H g(xh) dh = 1 \quad (x \in G) \quad \text{and}$$

$$(23) \quad \int_H g(xh)\omega(xh) dh \leq c\hat{\omega}([x]) \quad (x \in G).$$

Proof. We first construct a continuous function g_1 on G that satisfies

$$(24) \quad 0 < \int_H g_1(xh) dh < \infty \quad (x \in G) \quad \text{and}$$

$$(25) \quad \text{supp } g_1 \subset \{x \in G : \omega(x) \leq c\hat{\omega}([x])\},$$

where $c > 0$ is a constant.

Consider a non-trivial non-negative function $f \in C_{00}(G)$. Let

$$U = \{x \in G : f(x) > 0\}.$$

Then $U \neq \emptyset$ is an open set with a compact closure. Let $\tilde{c} > 0$ be a constant such that $\omega(u), \omega(u^{-1}) \leq \tilde{c}$ for every $u \in U$. (The existence of such \tilde{c} is justified by the compactness of \overline{U} and the continuity of ω .) We set $c = 2\tilde{c}^2$.

By Lemma 4.3, there exists a set $Y \subset G$ such that the family $\{UyH\}_{y \in Y}$ covers G and is locally finite. For every $y \in Y$, by the definition of $\hat{\omega}$, there is $y_0 \in [y]$ such that $\omega(y_0) \leq 2\hat{\omega}([y])$. We define $g_{1,y}(x) = f(xy_0^{-1})$ ($x \in G$). Clearly, $g_{1,y} \geq 0$ is a continuous function with compact support, and

$$\{x : g_{1,y}(x) \neq 0\} = \{x : f(x) > 0\} \cdot y_0 = Uy_0 \subset UyH.$$

We now show that $g_{1,y}$ satisfies (25), which is equivalent to

$$(26) \quad Uy_0 \subset \{x \in G : \omega(x) \leq c\hat{\omega}([x])\}.$$

In fact, for each $u \in U$, by the choice of y_0 we have

$$\begin{aligned} \omega(uy_0) &\leq \omega(u)\omega(y_0) \leq 2\tilde{c}\hat{\omega}([y]) = 2\tilde{c} \inf_{h \in H} \omega(y_0h) \\ &\leq 2\tilde{c}\omega(u^{-1}) \inf_{h \in H} \omega(uy_0h) \leq 2\tilde{c}^2\hat{\omega}([uy_0]). \end{aligned}$$

So (26) holds. Next we prove that $g_{1,y}$ satisfies

$$(27) \quad 0 < \int_H g_{1,y}(xh) dh < \infty \quad (x \in UyH).$$

By definition, $g_{1,y}$ is a non-negative continuous function with a compact support. So the upper inequality holds. Since H is a normal subgroup of G , when $x \in UyH$ we have $xy_0^{-1} \in UH$, and hence there is $h_0 \in H$ such that $xy_0^{-1}h_0 \in U$. Because U is open, there is a non-trivial open subset V of H such that $xy_0^{-1}V \subset U$. Let $V_0 = y_0^{-1}Vy_0$. Then $V_0 \neq \emptyset$ is an open subset of H such that $xV_0y_0^{-1} \subset U$. Since $f > 0$ on U , $g_{1,y} > 0$ on xV_0 . Therefore,

$$\int_H g_{1,y}(xh) dh \geq \int_{V_0} g_{1,y}(xh) dh > 0.$$

Now we let

$$g_1 = \sum_{y \in Y} g_{1,y}$$

Note that since $\{x : g_{1,y}(x) \neq 0\} \subset UyH$ ($y \in Y$) and the family $\{UyH\}_{y \in Y}$ is locally finite, the sum in the definition of g_1 has only finitely many non-zero terms in a neighborhood of every point. This implies that g_1 is well-defined, and

because each $g_{1,y}$ is continuous, g_1 is also continuous on G . From (27) and the local finiteness of $\{UyH\}_{y \in Y}$ it follows that (24) holds. The inclusion (25) also holds since it holds for each $g_{1,y}$. So the function g_1 satisfies all our requirements.

We then define the function g by

$$g(x) = \frac{g_1(x)}{\int_H g_1(xh) dh} \quad (x \in G).$$

Clearly, g is a continuous non-negative function on G and it satisfies

$$\int_H g(xh) dh = \int_H \frac{g_1(xh)}{\int_H g_1(xht) dt} dh = \frac{\int_H g_1(xh) dh}{\int_H g_1(xt) dt} = 1 \quad (x \in G).$$

So (22) is satisfied. Moreover, it follows directly from (25) and (22) that

$$\int_H g(xh)\omega(xh) dh \leq c\hat{\omega}([x]) \int_H g(xh) dh = c\hat{\omega}([x]).$$

So (23) is also satisfied. The proof is complete. \square

Let g be a function ensured in Lemma 4.4 and T be the homomorphism given by (21). Define

$$(28) \quad (Pf)(x) = (Tf)([x])g(x) \quad (x \in G, f \in L^1(G, \omega)).$$

Then for each $f \in L^1(G, \omega)$, the function $P(f)$ is clearly measurable. By Weil's Formula and inequality (23) we have

$$\begin{aligned} \int_G |(Pf)(x)|\omega(x) dx &= \int_{G/H} \int_H |(Tf)([x])|g(xh)\omega(xh) dh d[x] \\ &= \int_{G/H} |(Tf)([x])| \int_H g(xh)\omega(xh) dh d[x] \\ &\leq \int_{G/H} |(Tf)([x])| \cdot c\hat{\omega}([x]) d[x] = c\|Tf\|_{1,\hat{\omega}} \leq c\|f\|_{1,\omega}. \end{aligned}$$

So $P : L^1(G, \omega) \rightarrow L^1(G, \omega)$ is a bounded operator with $\|P\| \leq c$.

Theorem 4.5. *Let G be a locally compact group, H be a closed normal subgroup of G , and ω be a weight on G bounded away from zero. Then the mapping $P : L^1(G, \omega) \rightarrow L^1(G, \omega)$ defined by (28) is a continuous projection whose kernel is $J_\omega(G, H)$.*

Proof. Obviously, $\ker(P) = \ker(T) = J_\omega(G, H)$. So we only need to verify that $P^2 = P$. In fact,

$$\begin{aligned} (P^2f)(x) &= (P(Pf))(x) = (T(Pf))([x])g(x) = \left(\int_H (Pf)(xh) dh \right) g(x) \\ &= g(x) \int_H (Tf)([xh])g(xh) dh = g(x)(Tf)([x]) \int_H g(xh) dh \\ &= (Tf)([x])g(x) = (Pf)(x) \quad (x \in G, f \in L^1(G, \omega)). \end{aligned}$$

Therefore, P is a projection. The proof is complete. \square

We do not know whether $J_\omega(G, H)$ has the trace extension property in general. The next lemma provides a sufficient condition for a complemented ideal to have the trace extension property.

Lemma 4.6. *Let A be a Banach algebra and I be a closed complemented ideal in A . Denote by I_0 the closure of*

$$\text{lin}\{at - ta : a \in A, t \in I\}.$$

Suppose that $A = I \oplus X$, where X is a closed subspace of A such that

$$xy - yx \in I_0 \oplus X \quad (x, y \in X).$$

Then I has the trace extension property.

Remark 4.7. There are two important special cases for which conditions of Lemma 4.6 are satisfied:

1. the complement X of I is a subalgebra of A ;
2. the complement X is commutative, i.e., $xy = yx$ for all $x, y \in X$ (note that xy may not be in X). In particular, this is the case if A is Abelian.

Our Lemma 4.6 generalizes [17, Lemma 2.3], where only the first case was concerned.

Proof of Lemma 4.6. Let $\lambda \in I^*$ satisfy $\lambda \cdot a = a \cdot \lambda$ ($a \in A$). The condition really means $\lambda(ta) = \lambda(at)$ for all $t \in I$ and $a \in A$, or, equivalently, $\lambda|_{I_0} = 0$. Since $A = I \oplus X$, we have that $A^* = I^* \oplus X^*$. We show that $\tau = \lambda \oplus 0$ is a trace extension of λ . Obviously, τ is a continuous linear functional on A , $\tau|_I = \lambda$, and $\tau|_{I_0 \oplus X} = 0$. Now let $a, b \in A$ such that $a = t_1 + x_1$ and $b = t_2 + x_2$ with $t_1, t_2 \in I$ and $x_1, x_2 \in X$. We have $\lambda(t_1b) = \lambda(bt_1)$ and $\lambda(t_2x_1) = \lambda(x_1t_2)$. So

$$\tau(ab) = \lambda(t_1b + x_1t_2) + \tau(x_1x_2) = \lambda(bt_1 + t_2x_1) + \tau(x_1x_2) = \tau(ba) + \tau(x_1x_2 - x_2x_1).$$

Since $x_1x_2 - x_2x_1 \in I_0 \oplus X$ by the assumption, $\tau(x_1x_2 - x_2x_1) = 0$. Therefore, $\tau(ab) = \tau(ba)$. This completes the proof. \square

Combining Theorem 4.5 with Proposition 4.2 and Lemma 4.6, we obtain the following.

Proposition 4.8. *Let G be a locally compact group, H be a closed normal subgroup of G , and ω be a weight on G bounded away from zero. Suppose that X is a Banach space complement of $J_\omega(G, H)$ in $L^1(G, \omega)$ such that*

$$xy - yx \in J_0 \oplus X \quad (x, y \in X),$$

*where J_0 is the closure of $\text{lin}\{f * j - j * f : f \in L^1(G, \omega), j \in J_\omega(G, H)\}$. Then weak amenability of $L^1(G, \omega)$ implies weak amenability of $L^1(G/H, \hat{\omega})$.*

We now consider the special case when $G = G_1 \times G_2$, $H = G_2$, and $\omega = \omega_1 \times \omega_2$ with ω_i bounded away from zero on G_i ($i = 1, 2$). In this case $G/H = G_1$,

$$\hat{\omega}(x_1) = \omega_1(x_1) \inf_{x_2 \in G_2} \omega_2(x_2) = \text{const} \cdot \omega_1(x_1),$$

and the operator $T : L^1(G, \omega) \rightarrow L^1(G/H, \hat{\omega}) \cong L^1(G_1, \omega_1)$ is precisely given by

$$T(f)(x_1) = \int_{G_2} f(x_1, x_2) dx_2 \quad (x_1 \in G_1).$$

Consider a non-negative function $h \in C_{00}(G_2)$ such that

$$\int_{G_2} h(x_2) dx_2 = 1.$$

Then $g(x_1, x_2) = h(x_2)$ satisfies

$$\int_{G_2} g(x_1, x_2) dx_2 = \int_{G_2} h(x_2) dx_2 = 1,$$

$$\int_{G_2} g(x_1, x_2) \omega(x_1, x_2) dx_2 = \omega_1(x_1) \int_{G_2} h(x_2) \omega_2(x_2) dy = \text{const} \cdot \hat{\omega}(x_1) \quad (x_1 \in G_1).$$

Note that $L^1(G, \omega) = L^1(G_1, \omega_1) \hat{\otimes} L^1(G_2, \omega_2)$, and so we have

$$(29) \quad J_\omega(G, H) = L^1(G_1, \omega_1) \hat{\otimes} I_2, \quad X = L^1(G_1, \omega_1) \hat{\otimes} (\mathbb{C}h),$$

where

$$I_2 = \left\{ f \in L^1(G_2, \omega_2) : \int_{G_2} f(x_2) dx_2 = 0 \right\}.$$

Proposition 4.9. *Let G_1, G_2 be locally compact groups and ω_i be a weight on G_i bounded away from zero ($i = 1, 2$). Suppose that $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$ is weakly amenable. Then both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$ are also weakly amenable.*

Proof. Because of the symmetry, it is enough to show that $L^1(G_1, \omega_1)$ is weakly amenable. For this case, as has been discussed,

$$L^1(G_1 \times G_2, \omega_1 \times \omega_2) = J_\omega(G, H) \oplus X$$

with $J_\omega(G, H)$ and X being given by (29).

For $f_1, f_2 \in L^1(G_1, \omega_1)$ we have

$$\begin{aligned} (f_1 \otimes h)(f_2 \otimes h) - (f_2 \otimes h)(f_1 \otimes h) &= (f_1 * f_2 - f_2 * f_1) \otimes (h * h) \\ &= (f_1 * f_2 - f_2 * f_1) \otimes (h * h - h) + (f_1 * f_2 - f_2 * f_1) \otimes h. \end{aligned}$$

The second term of the last expression belongs to X . We show that the first term belongs to J_0 . Denote $k = h * h - h$. It is easy to see that $k \in I_2$ and so $f_2 \otimes k \in J_\omega(G, H)$. Let (e_i) be a bounded approximate identity of $L^1(G_2, \omega_2)$. Then for each i

$$(f_1 \otimes e_i)(f_2 \otimes k) - (f_2 \otimes k)(f_1 \otimes e_i) \in J_0,$$

and hence

$$\begin{aligned} (f_1 * f_2 - f_2 * f_1) \otimes k &= \lim_i ((f_1 * f_2) \otimes (e_i * k) - (f_2 * f_1) \otimes (k * e_i)) \\ &= \lim_i ((f_1 \otimes e_i)(f_2 \otimes k) - (f_2 \otimes k)(f_1 \otimes e_i)) \in J_0. \end{aligned}$$

So we have shown that $(f_1 \otimes h)(f_2 \otimes h) - (f_2 \otimes h)(f_1 \otimes h) \in J_0 \oplus X$, and the condition of Proposition 4.8 holds. Thus, $L^1(G_1, \omega_1) \cong L^1(G/H, \hat{\omega})$ is weakly amenable if $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$ is weakly amenable. \square

5. BEURLING ALGEBRA OF SUBGROUPS

In spite of Proposition 4.9, weak amenability of $L^1(G_1 \times G_2, \omega)$ does not necessarily imply weak amenability of $L^1(G_1, \omega_1)$ even if the groups G_1, G_2 are commutative, where $\omega_1(x) = \omega(x, e_2)$ and e_2 is the unit of G_2 . We give a counterexample in the following.

Let G_1, G_2 be Abelian locally compact groups and $G = G_1 \times G_2$. Suppose that there exist continuous non-zero group homomorphisms $\Phi_i : G_i \rightarrow \mathbb{R}$ ($i = 1, 2$). For any $\alpha, \beta > 0$ we define the function ω on G as follows:

$$(30) \quad \omega(x, y) = (1 + |\Phi_1(x)|)^\alpha (1 + |\Phi_1(x) + \Phi_2(y)|)^\beta \quad (x \in G_1, y \in G_2).$$

It is readily seen that ω is a weight on G , and

$$\omega_1(x) = \omega(x, e_2) = (1 + |\Phi_1(x)|)^{\alpha+\beta} \quad (x \in G_1).$$

Example 5.1. Let G_1, G_2 , and ω be as above. If $0 < \alpha, \beta < 1/2$ and $\alpha + \beta \geq 1/2$, then $L^1(G, \omega)$ is weakly amenable, but $L^1(G_1, \omega_1)$ is not weakly amenable.

Proof. Since $\Phi_1 : G_1 \rightarrow \mathbb{R}$ is a non-trivial continuous group homomorphism and

$$\sup_{x \in G_1} \frac{|\Phi_1(x)|}{\omega_1(x)\omega_1(x^{-1})} = \sup_{x \in G_1} \frac{|\Phi_1(x)|}{(1 + |\Phi_1(x)|)^{2(\alpha+\beta)}} < \infty$$

if $\alpha + \beta \geq 1/2$, $L^1(G_1, \omega_1)$ is not weakly amenable due to Theorem 1.1. To show that $L^1(G, \omega)$ is weakly amenable, we consider any non-trivial continuous group homomorphism $\Phi : G \rightarrow \mathbb{R}$. We have

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \sup_{x \in G_1, y \in G_2} \frac{|\Phi(x, e_2) + \Phi(e_1, y)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_1(x) + \Phi_2(y)|)^{2\beta}}.$$

Case 1. If there is $y \in G_2$ such that $\Phi(e_1, y) \neq 0$, then

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} \geq \sup_{n \in \mathbb{N}} \frac{|\Phi(e_1, y^n)|}{\omega(e_1, y^n)\omega(e_1, y^{-n})} = \sup_{n \in \mathbb{N}} \frac{n\Phi(e_1, y)}{(1 + n|\Phi_2(y)|)^{2\beta}} = \infty,$$

since $\beta < 1/2$.

Case 2. If $\Phi(e_1, y) = 0$ for all $y \in G_2$, then we can choose $x_0 \in G_1$ such that $\Phi(x_0, e_2) \neq 0$. We can also choose $y \in G_2$ such that $\Phi_2(y) \neq 0$. For each $x \in G_1$, we take an $n = n(x) \in \mathbb{N}$ such that

$$\left| n + \frac{\Phi_1(x)}{\Phi_2(y)} \right| \leq 1.$$

It then follows that

$$|\Phi_1(x) + \Phi_2(y^n)| = |\Phi_1(x) + n\Phi_2(y)| \leq |\Phi_2(y)|.$$

Hence,

$$\begin{aligned} \sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} &\geq \sup_{x \in G_1} \frac{|\Phi(x, e_2)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_1(x) + \Phi_2(y^n)|)^{2\beta}} \\ &\geq \sup_{x \in G_1} \frac{|\Phi(x, e_2)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_2(y)|)^{2\beta}} \\ &\geq \sup_{m \in \mathbb{N}} \frac{m|\Phi(x_0, e_2)|}{(1 + m|\Phi_1(x_0)|)^{2\alpha} (1 + |\Phi_2(y)|)^{2\beta}} = \infty, \end{aligned}$$

because $\alpha < 1/2$.

So, we have shown that

$$\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \infty$$

for every non-trivial continuous group homomorphism $\Phi : G \rightarrow \mathbb{R}$. Therefore, $L^1(G, \omega)$ is weakly amenable by Theorem 1.1 (see [30, Theorem 3.5]). \square

Example 5.1 also shows that, unlike the group algebra case, in general weak amenability of a Beurling algebra on an Abelian group G does not imply weak amenability of the induced Beurling algebra on a subgroup of G . However, the implication is true for certain “large” open subgroups. We first give a technical lemma dealing with extension of a group homomorphism.

Lemma 5.2. *Let G be a locally compact Abelian group and H be an open subgroup of G . Then any continuous group homomorphism $\Phi : H \rightarrow \mathbb{C}$ can be extended to a continuous group homomorphism $\tilde{\Phi} : G \rightarrow \mathbb{C}$.*

Proof. By Zorn’s Lemma, it suffices to show that for every $g \in G$ we can extend Φ to the open subgroup $H_g = \bigcup_{n \in \mathbb{Z}} g^n H = \{g^n h : h \in H, n \in \mathbb{Z}\}$ of G .

Suppose first that there exists $m \in \mathbb{N}$ such that $g^m \in H$. Let m_0 be the smallest such number. Then we denote $\alpha = \frac{1}{m_0} \Phi(g^{m_0})$ and define $\tilde{\Phi}(g^n h) = n\alpha + \Phi(h)$ ($h \in H, n \in \mathbb{Z}$). It is easy to see that $\tilde{\Phi}$ is a group homomorphism on H_g . In fact, the only non-trivial assertion one needs to verify is that the extension is well-defined, i.e., if $g^{n_1} h_1 = g^{n_2} h_2$ then $n_1 \alpha + \Phi(h_1) = n_2 \alpha + \Phi(h_2)$. But in this case $g^{n_1 - n_2} = h_2 h_1^{-1} \in H$, and so $n_1 - n_2 = km_0$ for some $k \in \mathbb{Z}$. Because Φ is a group homomorphism on H , we then have

$$\Phi(h_2) - \Phi(h_1) = \Phi(h_2 h_1^{-1}) = \Phi(g^{n_1 - n_2}) = k\Phi(g^{m_0}) = km_0 \alpha = (n_1 - n_2)\alpha,$$

which implies the desired equality $n_1 \alpha + \Phi(h_1) = n_2 \alpha + \Phi(h_2)$. The extension $\tilde{\Phi}$ is also continuous on H_g . Indeed, let $\{t_\gamma = g^{n_\gamma} h_\gamma\}_{\gamma \in \Gamma} \subset H_g$ be a net that converges to some $t = g^n h \in H_g$. We have $g^{n_\gamma - n} h_\gamma \xrightarrow{\gamma} h$. Since H is open, there is $\gamma_0 \in \Gamma$ such that $g^{n_\gamma - n} \in H$ for $\gamma \geq \gamma_0$. Then from the continuity of Φ on H it follows that

$$\tilde{\Phi}(g^{n_\gamma - n} h_\gamma) = \Phi(g^{n_\gamma - n} h_\gamma) \xrightarrow{\gamma} \Phi(h) = \tilde{\Phi}(h).$$

Using the fact that $\tilde{\Phi}$ is a group homomorphism, we finally obtain

$$\tilde{\Phi}(t_\gamma) = \tilde{\Phi}(g^{n_\gamma} h_\gamma) = \tilde{\Phi}(g^{n_\gamma - n} h_\gamma) + \tilde{\Phi}(g^n) \xrightarrow{\gamma} \tilde{\Phi}(h) + \tilde{\Phi}(g^n) = \tilde{\Phi}(g^n h) = \tilde{\Phi}(t).$$

Now assume that $g^n \notin H$ for all $n \in \mathbb{N}$. Then we put $\tilde{\Phi}(g^n h) = \Phi(h)$ ($h \in H, n \in \mathbb{Z}$). Obviously, $\tilde{\Phi}$ is a group homomorphism on H_g . We now show that it is continuous. Let $g^{n_\gamma} h_\gamma \xrightarrow{\gamma} g^n h$ ($n_\gamma, n \in \mathbb{Z}, h_\gamma, h \in H$). Then, as above, $g^{n_\gamma - n} h_\gamma \xrightarrow{\gamma} h$ and, because H is open, there is γ_0 such that $g^{n_\gamma - n} \in H$ for $\gamma \geq \gamma_0$. But our assumption on g implies that this is possible only when $n_\gamma = n$ for $\gamma \geq \gamma_0$, and so $h_\gamma \xrightarrow{\gamma} h$. Therefore,

$$\tilde{\Phi}(g^{n_\gamma} h_\gamma) = \Phi(h_\gamma) \xrightarrow{\gamma} \Phi(h) = \tilde{\Phi}(g^n h).$$

This shows that $\tilde{\Phi}$ is continuous. The proof is complete. \square

In general, one cannot expect that a group homomorphism Φ from a normal subgroup H of G has an extension to the whole G . In fact, if such extension exists then Φ must satisfy $\Phi(ghg^{-1}) = \Phi(h)$ for all $g \in G$ and $h \in H$. It turns out that the latter condition is also sufficient for semidirect product group $G = L \ltimes H$, where H is a normal subgroup and L is a subgroup of G such that $L \cap H = \{e\}$.

Proposition 5.3. *Let $G = L \ltimes H$ and $\Phi : H \rightarrow \mathbb{R}$ be a group homomorphism. Then Φ extends to a group homomorphism $\tilde{\Phi} : G \rightarrow \mathbb{R}$ if and only if*

$$(31) \quad \Phi(lhl^{-1}) = \Phi(h) \quad (l \in L, h \in H).$$

Moreover, if H is open in G then $\tilde{\Phi}$ is continuous whenever Φ is continuous.

Proof. The necessity part is trivial.

For sufficiency, we note that every $g \in G$ may be uniquely expressed in the form $g = lh$. Suppose that (31) holds. We then extend Φ to $\tilde{\Phi}$ on the whole G simply by letting $\tilde{\Phi}(g) = \Phi(h)$ ($g = lh, l \in L, h \in H$). It is a group homomorphism because for any $g_1 = l_1h_1, g_2 = l_2h_2 \in G$ we have

$$\begin{aligned} \tilde{\Phi}(g_1g_2) &= \tilde{\Phi}(l_1h_1l_2h_2) = \tilde{\Phi}((l_1l_2)(l_2^{-1}h_1l_2h_2)) = \Phi(l_2^{-1}h_1l_2h_2) \\ &= \Phi(l_2^{-1}h_1l_2) + \Phi(h_2) = \Phi(h_1) + \Phi(h_2) = \tilde{\Phi}(g_1) + \tilde{\Phi}(g_2). \end{aligned}$$

Assume now that H is open in G and that Φ is continuous on H . Let $g_i = l_ih_i \rightarrow g = lh$ ($h_i, h \in H, l_i, l \in L$). Then $l^{-1}l_ih_i \rightarrow h \in H$. Since H is open, it follows that $l^{-1}l_ih_i \in H$ ($i \geq i_0$) for some i_0 . Then $l^{-1}l_i \in H \cap L$ and hence $l_i = l$ for $i \geq i_0$. This implies that $h_i \rightarrow h$. Using the continuity of Φ we finally obtain

$$\tilde{\Phi}(g_i) = \Phi(h_i) \rightarrow \Phi(h) = \tilde{\Phi}(g).$$

Therefore, $\tilde{\Phi}$ is also continuous. \square

Proposition 5.4. *Let G be a locally compact IN group and ω be a weight on it. Suppose that H is a commutative subgroup of G , and suppose that every continuous group homomorphism $\Phi : H \rightarrow \mathbb{C}$ can be extended to the whole G . If there is $c > 0$ such that for each $x \in G$ there is $k = k(x) \in \mathbb{N}$ for which $x^k \in H$ and*

$$(32) \quad \frac{\omega(x^k)\omega(x^{-k})}{k} \leq c\omega(x)\omega(x^{-1}),$$

then weak amenability of $L^1(G, \omega)$ implies weak amenability of $L^1(H, \omega|_H)$.

Remark 5.5. In particular, the conditions of Proposition 5.4 are satisfied when G/H is a torsion group (see [13, A.1]).

Proof of Proposition 5.4. If $L^1(H, \omega|_H)$ is not weakly amenable, by Theorem 1.1 there is a non-trivial continuous group homomorphism $\Phi : H \rightarrow \mathbb{C}$ such that

$$\sup_{h \in H} \frac{|\Phi(h)|}{\omega(h)\omega(h^{-1})} = r < \infty.$$

By our assumption, Φ can be extended to a continuous group homomorphism $\tilde{\Phi} : G \rightarrow \mathbb{R}$. We have

$$\frac{|\tilde{\Phi}(x)|}{\omega(x)\omega(x^{-1})} = \frac{|\Phi(x^k)|}{\omega(x^k)\omega(x^{-k})} \frac{\omega(x^k)\omega(x^{-k})}{k} \frac{1}{\omega(x)\omega(x^{-1})} \leq rc$$

since $x^k \in H$, where $k = k(x) \in \mathbb{N}$ is such that (32) is satisfied. Then, by Theorem 2.2, $L^1(G, \omega)$ is not weakly amenable.

□

Corollary 5.6. *Let G be a locally compact $[IN]$ group and H be a commutative subgroup of G of finite index. Suppose that each continuous group homomorphism from H to \mathbb{C} can be continuously extended to the whole G . Then, for every weight ω on G such that $L^1(G, \omega)$ is weakly amenable, $L^1(H, \omega|_H)$ is also weakly amenable.*

Proof. Suppose, to the contrary, that $L^1(H, \omega|_H)$ is not weakly amenable. Then, since H is commutative, Theorem 1.1 implies the existence of a non-trivial continuous group homomorphism $\Phi : H \rightarrow \mathbb{C}$ and a constant $c > 0$ such that

$$\frac{|\Phi(h)|}{\omega(h)\omega(h^{-1})} \leq c \quad (h \in H).$$

By the assumption Φ extends to a continuous group homomorphism $\tilde{\Phi} : G \rightarrow \mathbb{C}$. Because H is of finite index, there exist $g_1, g_2, \dots, g_n \in G$ such that $G = \cup_{i=1}^n g_i H$. Hence, every $g \in G$ can be written in the form $g = g_i h$ for some $1 \leq i \leq n$, $h \in H$, and so

$$\begin{aligned} \frac{|\tilde{\Phi}(g)|}{\omega(g)\omega(g^{-1})} &\leq \frac{|\tilde{\Phi}(h)| + |\tilde{\Phi}(g_i)|}{\omega(g_i h)\omega(h^{-1}g_i^{-1})} \leq \frac{|\tilde{\Phi}(h)| + |\tilde{\Phi}(g_i)|}{\omega(h)\omega(h^{-1})} \cdot \omega(g_i)\omega(g_i^{-1}) \\ &\leq \max_{1 \leq i \leq n} \left(c + |\tilde{\Phi}(g_i)| \right) \omega(g_i)\omega(g_i^{-1}) = \text{const.} \end{aligned}$$

It follows that $L^1(G, \omega)$ is not weakly amenable by Theorem 2.2, which contradicts our assumption. □

Given a locally compact group G and a closed normal subgroup H of it, we have seen that weak amenability of $L^1(G, \omega)$ does not pass to $L^1(H, \omega|_H)$ in general even G is commutative. One may wonder whether the condition that both $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ are weakly amenable forces $L^1(G, \omega)$ to be weakly amenable. It turns out that the answer is also negative. A counterexample is as follows.

Example 5.7. We consider $G = \mathbf{ax} + \mathbf{b}$ and $H = H_{\mathbf{b}}$. Suppose that w is a weight on (\mathbb{R}^+, \cdot) that is not diagonally bounded, but such that $\ell^1(\mathbb{R}^+, w)$ is weakly amenable. (For example, we can take $w(a) = (1 + |\ln a|)^\alpha$, $0 < \alpha < 1/2$.) We then define ω on $\mathbf{ax} + \mathbf{b}$ by $\omega(a, b) = w(a)$ ($a > 0$). Clearly, ω is a weight on G , $\omega|_H = \text{constant}$, and $\hat{\omega} = w$. So $\ell^1(H, \omega|_H)$ and $\ell^1((\mathbf{ax} + \mathbf{b})/H, \hat{\omega})$ are both weakly amenable. But by our assumption ω is not diagonally bounded, and so $\ell^1(\mathbf{ax} + \mathbf{b}, \omega)$ is not weakly amenable due to Proposition 3.2.

Even G is finitely generated, this situation could happen.

Example 5.8. Let $\mathbb{Z}[\frac{1}{2}]$ denote the set of all dyadic fractions, i.e., the set of all rational numbers whose binary expansion is finite. Consider the countable subgroup G_2 of $\mathbf{ax} + \mathbf{b}$ defined by

$$G_2 = \left\{ (2^n, b) : n \in \mathbb{Z}, b \in \mathbb{Z} \left[\frac{1}{2} \right] \right\}.$$

In fact, G_2 is the subgroup of $\mathbf{ax} + \mathbf{b}$ generated by the elements $(2, 0)$ and $(1, 1)$, and so it is a finitely generated amenable group. Let

$$H_2 = H_{\mathbf{b}} \cap G_2 = \left\{ (1, b) : b \in \mathbb{Z} \left[\frac{1}{2} \right] \right\}.$$

Then H_2 is a normal subgroup of G_2 and $G_2/H_2 \cong (\mathbb{Z}, +)$. On G_2 we consider the weight ω_α ($0 < \alpha < 1/2$) defined by

$$\omega_\alpha(2^n, b) = (1 + |n|)^\alpha \quad (n \in \mathbb{Z}).$$

The same argument as in Example 5.7 shows that $\ell^1(G_2, \omega_\alpha)$ is not weakly amenable while both $\ell^1(H_2, \omega_\alpha)$, which is isomorphic to $\ell^1(H_2)$, and $\ell^1(G_2/H_2, \hat{\omega}_\alpha)$, which is isometrically isomorphic to $\ell^1(\mathbb{Z}, \omega_\alpha)$, are weakly amenable. We are grateful to N. Spronk for this observation.

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